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Symmetries in a Constrained System with a Singular Higher-Order Lagrangian

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A simple algorithm to construct the generator of gauge transformation for a constrained canonical system with a singular higher-order Lagrangian in field theories is developed. Based on phase-space generating functional of Green function for such a system, the generalized canonical Ward identities under the non-local transformation have been deduced. For the gauge-invariant system, based on configuration-space generating functional, the generalized Ward identities under the non-local transformation have been also derived.The conservation laws are deduced at the quantum level. The applications of the above results to the gauge invariance massive vector field and non-Abelian Chern–Simons(CS) theories with higher-order derivatives are given, a new form of gauge-ghost proper vertices, and Ward–Takahashi identity under BRS transformation and BRS charge at the quantum level are obtained. In the canonical formulation one does not need to carry out the integration over canonical momenta in phase-space path integral as usually performed.

KEY WORDS: symmetries; constrained system; singular higher-order Langrian.

1. INTRODUCTION

Local symmetry is a central concept in modern field theories. The connection between continuous symmetries and conservation laws are usually referred to as Noether's theorem in classical theories. Classical Noether theorems and their generalization have been formulated in terms of Lagrange's variables in configuration space (Li, 1993a). Recently, the canonical symmetry in classical theories has been performed (Li, 1993a,b; Deriglazov and Evdokimov, 2000). The classical second

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Noether theorem or Noether identity refers to invariance of the action integral of the system under a local transformation parameterized by r arbitrary functions and their derivatives. In quantum theory, the Noether identity corresponds to the Ward (or Ward–Takahashi) identity.

Dynamical systems described in terms of higher-order derivative Lagrangians have close relations with relative particle dynamics, gravity, gauge theories, modified KdV equations, supersymmetry, string model and other problems (Li, 1993a,b; Deriglazov and Evdokimov, 2000), and it has attracted much attention recently (Borneas and Damian, 1999; Damian, 2000).

As is well known, the identities relating the Green function in QED were obtained by Ward (1950) and Takahashi (1957). In the non-Abelian theories, their role is played by the so-called generalized Ward identities, first obtained by Slavnov (1972) and Taylor (1971). Ward identities (or Ward–Takahashi identities) and their generalization play an important role in modern field theories. It has been used to prove renormalization for the theories of field and applied to calculate practical problem, for example, the calculation of higher-order proper vertices can be reduced to the lower-order proper vertices in QCD with the aid of such identities. Ward identities have been generalized to the supersymmetry (Joglekar, 1991) and superstring theories (Danilov, 1991) and other problems. The traditional derivations for the Ward identities in the functional integration method are usually discussed in configuration space (Suura and Young, 1973; Young, 1987; Lhallabi, 1989), which is valid for the case when the integration in the phase-space path integral over the canonical momenta belongs to the Gauss-type category. Phase-space path integrals are more basic than configuration-space path integrals, the latter provide for a Hamiltonian quadratic in the canonical momenta, whereas the former apply to arbitrary Hamiltonian. Thus, phase-space form of the path integral is a necessary precursor to the configuration-space form of the path integral (Mizrahi, 1978). For the certain case, while the phase-space path integral can be simplified by carrying out explicit integration over canonical momenta in which the "mass" depends on coordinates (Lee and Yang, 1962; Gerstein *et al.*, 1971) or depends on coordinates and momenta (Du *et al.*, 1980), the effective Lagrangians in configuration space are singularities with delta-function. For the constrained canonical (Hamiltonian) system with complicated constraints, especially, for the system with singular higher-order Lagrangian, it is very difficult or even impossible to carry out the integration over momenta (Li, 1994b). Therefore, the investigation of symmetry properties of the system in the phase space has more basic sense. Based on the invariance of the phase-space generating functional of Green's function for a system with singular first-order Lagrangian under the local transformation of canonical variables in extended phase space, the canonical Ward identities for such a system has been examined in a previous work (Li, 1995, 1999), for a system with a singular higher-order Lagrangian this problem a brief discussion has been also performed (Li, 1994a). Here the symmetries in a constrained canonical system with a singular higher-order Lagrangian will be further investigated in detail.

The paper is organized as follows. In Section 2, we develop a simple algorithm to construct the generator of the gauge transformation for a system with a singular higher-order Lagrangian in field theories, in which the series of first-class constraints derived from primary first-class constraints are completely separated from the series of the second-class ones. Once the canonical Hamiltonian and first-class constraints of the theory are given, the generator of the gauge transformation can be constructed. In Section 3, based on phase-space generating functional of the Green function for a system with a singular higher-order Lagrangian, the generalized canonical Ward identities under the non-local transformation of the canonical variables in extended phase space have been deduced. In this formulation we do not need to carry out the integration over canonical momenta in phase-space path integral. In Section 4, for the gauge-invariant system, based on configuration-space generating functional obtained by using Faddeev–Popov trick, the generalized Ward identities in configuration space have been also derived under the non-local transformation. In Section 5, the quantal conservation laws are deduced under the global symmetry transformation in phase space. In Section 6, we give an application of aforementioned results to the gauge invariance massive vector field with higher-order derivatives, the gauge generator have been constructed and some Ward identities for proper vertices are also derived. In Section 7, the application to non-Abelian CS theories with higher-order derivatives is given, a new form of gauge-ghost proper vertices have been derived under the non-local transformation. The Ward–Takahashi identity under the Beechi–Rouet–Stora (BRS) transformation is deduced and BRS conserved quantity at the quantum level is also obtained.

2. GAUGE GENERATORS

Let us consider a physical field defined by n field functions $\phi^\alpha(x)$ $(\alpha = 1, 2, \ldots, n)$, $x = (x_0, x_i)$ $(x_0 =$ t, i = 1, 2, 3), $g_{\mu\nu} = (1 - 1 - 1 - 1)$, and the motion of the field described by a Lagrangian involving high-order derivatives in the form of a functional,

$$L = \int d^3x L\big(\phi, \phi,_\mu, ..., \phi,_{\mu(N)}\big) \tag{2.1}$$

where $\phi,_\mu = \partial_\mu\phi = \frac{\partial}{\partial x^\mu}\phi$, $\quad \phi,_{\mu(m)} = \underbrace{\partial_\mu ... \partial_\sigma}_{m} \phi$, We denote $\phi_{(0)} = \phi$, $\phi_{(1)} = \dot{\phi}$, $\phi_{(2)} = \ddot{\phi}$, and using the Ostrogradsky transformation, we introduce the generalized canonical momenta

$$\pi_{\alpha}^{(s-1)} = \sum_{j=0}^{N-s} (-1)^j \frac{d^j}{dt^j} \frac{\delta L}{\delta \phi_{(j+s)}^{\alpha}} \tag{2.2}$$

and using these relations we can go over from the Lagrangian description to Hamiltonian description. The canonical Hamiltonian is defined by

$$H_C \left[\phi_{(s)}^{\alpha}, \pi_{\alpha}^{(s)}\right] = \int d^3 x H_C = \int d^3 x \left(\pi_{\alpha}^{(s)} \phi_{(s+1)}^{\alpha} - L\right) \tag{2.3}$$

which may be formed by eliminating only the highest derivatives $\phi_{(N)}$. The summation over indices α from 1 to n, s from 1 to N is taken repeatedly. For the singular Lagrangian L, the generalized Hessian matrix $(H_{\alpha\beta})$ is degenerate, $\det|H_{\alpha\beta}| = \det|\delta^2 L/\delta\phi_{(N)}^{\alpha}\delta\phi_{(N)}^{\beta}| = 0 = 0$. Hence, one cannot solve all $\phi_{(N)}^{\alpha}$ from the definition of the canonical momenta. Then there are constraints among the canonical variables in phase space (Saito *et al.*, 1989)

$$\Phi_a^0 \left(\phi_{(s)}^{\alpha}, \pi_{\alpha}^{(s)}\right) \approx 0 \quad (a = 1, 2, \ldots, n - R) \tag{2.4}$$

where the sign $\approx$ (weak equality) means equality on the constrained hypersurface, the rank of generalized Hessian matrix is assumed to be R. Equation (2.4) is called the primary constraint (PC). Thus, a system with a singular higher-order Lagrangian is subject to some inherent phase-space constraints and is called a generalized constrained canonical system. The generalized canonical equations for this generalized constrained canonical system can be written as (Saito *et al.*, 1989)

$$\dot{\phi}_{(s)}^{a} \approx \left\{\phi_{(s)}^{a}, H_T\right\}, \quad \dot{\pi}_{a}^{(s)} \approx \left\{\pi_{a}^{(s)}, H_T\right\}, \tag{2.5}$$

where

$$H_T = \int_V d^3 x \left(\mathcal{H}_C + \lambda^a \Phi_a^0\right) \tag{2.6}$$

$\lambda^a(x)$ are the Lagrange (constraint) multipliers, and $\{\cdot,\cdot\}$ denotes generalized Poisson bracket. Following the Dirac theory of constrained system, from the stationary conditions of primary constraints, one can define successively the secondary constraints,

$$\Phi_a^k = \left\{\Phi_a^{k-1}, H_T\right\} \approx 0 \tag{2.7}$$

This algorithm is continued until Φ_a^m satisfy

$$\Phi_a^{m+1} = \left\{\Phi_a^m, H_T\right\} = C_{ak}^b \Phi_b^k \quad (k \leq m) \tag{2.8}$$

All the constraints $\{\Psi_k\}$ are classified into two classes, Ψ_a are defined to be the first-class ones if $\{\Psi_a,\ \Psi_b\} = 0 (\mathrm{mod}\, \Psi_c)$ for all Ψ_b, otherwise they are the second-class ones.

The construction of the generator of local gauge transformation (including higher-order derivatives theories) had been studied by many authors. Following the prescription of Dirac, an ansatz was presented that the generator of gauge transformation for first-order derivatives theories is given as a linear combination of the first-class constraints of a constrained canonical system, where the gauge parameters are allowed to depend in general on time, as well as on the phase-space variables and Lagrange (constraint) multipliers (Banerjee *et al.*, 1999, 2000a,b). For some physically interesting models, these gauge parameters can be taken to be a function of time only. In this case, the ansatz for the generator of gauge transformation was proposed by earlier approches (Castellani, 1982; Shizad and Moghadam, 1999; Garcia and Pons, 2000), and the generator was constructed, and the applications to Yang–Mills theories and other problems were given. The restruction on gauge parameters depending on time only was also discussed in an earlier work (Galvão and Boechat, 1990), and the result agrees with the conclusion given by Castellani (Li, 1995). One of the authors had used the Castellani's ansatz to construct the generator of gauge transformation for second-order derivatives system with finite degrees of freedom (Li, 1991). We present here a similar algorithm to construct the generator of gauge transformation for a system with a singular higher-order Lagrangian in field theories.

For the sake of simplicity, all the constraints of the system are assumed to be of first class. It is supposed that constraint conditions (2.4) and equations of motion (2.5) are invariant under the infinitesmall change of the variables

$$\phi^{\alpha'}_{(s)}(x) = \phi^{\alpha}_{(s)}(x) + \xi^{\alpha}_{(s)}(x), \quad \pi^{(s)'}_{\alpha}(x) = \pi^{(s)}_{\alpha}(x) + \eta^{(s)}_{\alpha}(x), \quad \lambda^{a'}(x) = \lambda^{a}(x) + \zeta^{a}(x) \tag{2.9}$$

i.e. the unvaried trajectory $(\phi^{\alpha}_{(s)}, \pi^{(s)}_{\alpha}, \lambda^{a})$ and varied trajectory $(\phi^{\alpha'}_{(s)}, \pi^{(s)'}_{\alpha}, \lambda^{a'})$ both satisfy Equations (2.4) and (2.5), and let the generator of infinitesimal gauge transformation is denoted by G, then

$$\xi^{\alpha}_{(s)} = \delta\phi^{\alpha}_{(s)} = \left\{\phi^{\alpha}_{(s)}, G\right\} = \frac{\delta G}{\delta \pi^{(s)}_{\alpha}} \tag{2.10a}$$

$$\eta^{(s)}_{\alpha} = \delta\pi^{(s)}_{\alpha} = \left\{\pi^{(s)}_{\alpha}, G\right\} = -\frac{\delta G}{\delta \phi^{\alpha}_{(s)}} \tag{2.10b}$$

Following Castellani's ansatz, we consider the gauge generator of type

$$G(\varepsilon, \varepsilon, \ddot{\varepsilon}, \ldots) = \int d^3x \varepsilon^{(k)}_{j} G^{j}_{k}\left(\phi^{\alpha}_{(s)}, \pi^{(s)}_{\alpha}\right) \quad \left(\varepsilon^{(k)}_{j} = \partial^{k}_{0}\varepsilon_{j}(x),\ k = 0, 1, \ldots, K_1\right) \tag{2.11}$$

where $\varepsilon_j(x)$ $(j = 1, 2, \ldots, J_1)$ are the arbitrary functions. We substitute (2.10) into Equations (2.4) and (2.5) of varied trajectory and expand it to first order in the small variations and using Equations (2.4) and (2.5) for unvaried trajectory, as

$\varepsilon_j(x)$ are arbitrary, we obtain

$$\{G_k^j, \Phi_a^0\} = 0 \quad (\text{mod PC}) \tag{2.12}$$

$$\frac{\partial}{\partial \phi_{(s)}^\alpha}\left(G_{k-1}^j + \{G_k^j, H_T\}\right) = 0 \quad (\text{mod PC}) \tag{2.13a}$$

$$\frac{\partial}{\partial \pi_\alpha^{(s)}}\left(G_{k-1}^j + \{G_k^j, H_T\}\right) = 0 \quad (\text{mod PC}) \tag{2.13b}$$

Because we are considering the variation that leave the trajectory on constrained hypersurface, for secondary constraints Φ_a^n, one should add the further requirements $\{G_k^j, \Phi_a^n\} = 0$, hence all G_k^j have to be first-class constraints. In the expression (2.13) H_C, can be substituted instead of H_T , owing to the assumption that all constraints are first-class ones, it follows

$$G_m^j = 0 \quad (\text{mod PC}) \tag{2.14}$$

$$G_{k-1}^j + \{G_k^j, H_C\} = 0 \quad (\text{mod PC}) \tag{2.15}$$

$$\{G_0^j, H_C\} = 0 \quad (\text{mod PC}) \tag{2.16}$$

Therefore, all the G_k^j have to be first-class constraints, G_{k-1}^j is deduced from G_k^j according to the recursive relations (2.15). All the G_k^j have to be first-class constraints, with the exception of those first-class constraints which arise as powers λ^n, are part of the gauge generations (Castellani, 1982; Shizad and Moghadam, 1999; Garcia and Pons, 2000). Moreover, G_m^j must be a primary first-class constraint, from every primary first-class constraint, according to (2.15) constructing the chain of G_k^j until G_0^j is reached. Using (2.11) one obtains the generator of gauge transformation.

The recursive relations (2.15) are just a consequence of the conservation law G, as from (2.11) it follow

$$\dot{G} = \frac{\partial G}{\partial t} + \{G, H_T\} = \int d^3x \varepsilon_j^{(k)}\left(G_{k-1}^j + \{G_k^j, H_T\}\right) = 0 \ (\text{mod PC}). \tag{2.17}$$

Using the so-called master equation (Banerjee *et al.*, 1999; Banerjee *et al.*, 2000a,b), one can also give these results. H_C can be also substituted instead of H_T and $\varepsilon_j^{(x)}$ are arbitrary, then from (2.17) we get the expressions (2.15).

Even when second-class constraints appear, if the series of first-class constraints derived from the primary first-class constraints are completely separated from the series of the second-class ones, this formulation to construct the gauge generator is valid for such a system.

3. GENERALIZED CANONICAL WARD IDENTITIES

Let us consider a system with a singular higher-order Lagrangian, this system is subject to some inherent phase-space constraints. Let $\Lambda_k(\phi^\alpha_{(s)}, \pi^{(s)}_\alpha) \approx 0$ $(k = 1, 2, ..., K)$ be first-class constraints, and $\theta_i(\phi^\alpha_{(s)}, \pi^{(s)}_\alpha) \approx 0$ $(i = 1, 2, ..., I)$ be second-class constraints. According to the rule of path integral quantization, for each first-class constraint, one must choose a gauge condition. The phase-space generating functional of Green's function for a system with a singular higher-order Lagrangian can be written as (Gitman and Tyutin, 1990)

$$Z[j,k] = \int \mathcal{D}\phi^\alpha_{(s)}\mathcal{D}\pi^{(s)}_\alpha \delta(\Phi)\sqrt{\det\{\Phi,\Phi\}}\exp \times \left\{ i\int d^4x \left[\pi^{(s)}_\alpha \phi^\alpha_{(s+1)} - \mathcal{H}_C + j^s_\alpha\phi^\alpha_{(s)} + k_s\pi^{(s)}_\alpha\right]\right\} \tag{3.1}$$

where $\mathcal{H}_C$ is a canonical Hamiltonian density, $\Phi = \{\Phi_{n'}\}$ is a set of all constraints (for a theory with second-class constraints) or the set of constraints and gauge conditions (for a theory with first-class constraints), j^s_α and k^α_s are exterior sources with respect to $\phi^\alpha_{(s)}$ and $\pi^{(s)}_\alpha$, respectively. By making use of the properties of the δ-function and the integration over Grassmann variables $C_l(x)$ and $\bar{C}_k(x)$, one gets

$$Z[j,k] = \int \mathcal{D}\phi^\alpha_{(s)}\mathcal{D}\pi^{(s)}_\alpha\mathcal{D}\lambda_m\mathcal{D}\bar{C}_k\mathcal{D}C_l \exp \times \left\{ i\int d^4x \left[\mathcal{L}^P_{\text{eff}} + j^s_\alpha\phi^\alpha_{(s)} + k_s\pi^{(s)}_\alpha\right]\right\} \tag{3.2}$$

where

$$L^P_{\text{eff}} = \pi^{(s)}_\alpha\phi^\alpha_{(s+1)} - \mathcal{H}_C + \lambda_{n'}\Phi_{n'} + \frac{1}{2}\int d^4y\bar{C}_k(x)\{\Phi_k(x), \Phi_l(y)\}C_l(y) \tag{3.3}$$

and $\lambda_{n'}(x)$ are the Lagrange multipliers.

For the sake of simplicity, we put $\phi_{(s)} = (\phi^\alpha_{(s)}, \lambda_{n'}, \bar{C}_k, C_l)$, $J^s = (j^s_\alpha, \eta_{n'}, \xi_k, \bar{\xi}_l)$, where $\eta_{n'}$, ξ_k and $\bar{\xi}_l$ are the exterior sources with respect to $\lambda_{n'}$, $\bar{C}_k$ and C_l, respectively, and $\pi^{(s)} = (\pi^{(s)}_\alpha)$, $K_s = (k^\alpha_s)$, then the expression (3.2) can be written as

$$Z[J,K] = \int \mathcal{D}\phi_{(s)}\mathcal{D}\pi^{(s)}\exp\left\{i\int d^4x\left[\mathcal{L}^P_{\text{eff}} + J^s\phi_{(s)} + K_s\pi^{(s)}\right]\right\} \tag{3.4}$$

Local invariance play an important role in the gauge field theories, and non-local transformation in field theories had been also introduced (Kuang and Yi, 1980; Fradkin and Palchik, 1984). Now we consider the transformation properties of the phase-space generating functional under general local and non-local transformation with the following form of infinitesimal transformation in extended phase space

$$\begin{cases} x^{\mu'} &= x^\mu + \Delta x^\mu = x^\mu + R^\mu_\sigma \varepsilon^\sigma(x) \\ \phi_{(s)'}(x') &= \phi_{(s)}(x) + \Delta\phi_{(s)}(x) = \phi_{(s)}(x) + A_{s\sigma}\varepsilon^\sigma(x) + \int d^4x E(x,y) B_{s\sigma}\varepsilon^\sigma(y) \\ \pi^{(s)'}(x') &= \pi^{(s)}(x) + \Delta\pi^{(s)}(x) = \pi^{(s)}(x) + U^s_\sigma \varepsilon^\sigma(x) + \int d^4x F(x,y) V^s_\sigma \varepsilon^\sigma(y) \end{cases} \tag{3.5}$$

where $E(x,y)$ and $F(x,y)$ are some functions, and R^μ_σ, $A_{s\sigma}$, $B_{s\sigma}$, U^s_σ and V^s_σ are the linear differential operators,

$$\begin{aligned} &R^\mu_\sigma = r^{\mu(l)}_\sigma \partial_{\mu(l)} \quad A_{s\sigma} = a^{(m)}_{s\sigma}\partial_{(m)} \quad B_{s\sigma} = b^{(n)}_{s\sigma}\partial_{(n)} \quad U^s_\sigma = u^{s(p)}\partial_{(p)} \\ &V^s_\sigma = v^{s(q)}\partial_{(q)} \quad r^{\mu(l)}_\sigma = r_\sigma^{\overbrace{\mu\nu\ldots\lambda}^{l}} \quad \partial_{\mu(l)} = \underbrace{\partial_\mu\partial_\nu\ldots\partial_\lambda}_{l} \quad \text{etc.} \end{aligned} \tag{3.6}$$

where $r^{\mu(l)}_\sigma$, $a^{(m)}_{s\sigma}$, $b^{(n)}_{s\sigma}$, $u^{s(p)}$ and $v^{s(q)}$ are some functions of x, $\phi_{(s)}$ and $\pi^{(s)}$, $\varepsilon^\sigma(x)$ ($\sigma = 1,2\ldots,s$) are infinitesimal arbitrary functions, and their values and derivatives up to required order are vanishing on the boundary of the space–time domain. The variation of an effect canonical action (3.3) under the transformation (3.5) is given by (Li, 1999a)

$$\Delta I^P_{\text{eff}} = \int d^4x \left\{ \frac{\delta I^P_{\text{eff}}}{\delta\phi_{(s)}}\delta\phi_{(s)} + \frac{\delta I^P_{\text{eff}}}{\delta\pi^{(s)}}\delta\pi^{(s)} \right. \\ \left. + \partial_\mu[(\pi^{(s)}\phi_{(s)} - \mathcal{H}_{\text{eff}}\Delta x^\mu] + D(\pi^{(s)}\delta\phi_{(s)}) \right\}, \tag{3.7}$$

where $D = \frac{d}{dt}$, and

$$\frac{\delta I^P_{\text{eff}}}{\delta\phi_{(s)}} = -\dot{\pi}^{(s)} - \frac{\delta H_{\text{eff}}}{\delta\phi_{(s)}}, \quad \frac{\delta I^P_{\text{eff}}}{\delta\pi^{(s)}} = \dot{\phi}_{(s)} - \frac{\delta H_{\text{eff}}}{\delta\pi^{(s)}} \tag{3.8}$$

$$\delta\phi_{(s)} = \Delta\phi_{(s)} - \phi_{(s)'_\mu}\Delta x^\mu, \quad \delta\pi^{(s)} = \Delta\pi^{(s)} - \pi^{(s)}_{,\mu}\Delta x^\mu, \tag{3.9}$$

The Jacobian of the transformation (3.5) is denoted by $\bar{J}[\phi_{(s)}, \pi^{(s)}, \varepsilon]$. The generating functional (3.4) is invariant under the transformation (3.5), which implies that $\delta Z / \delta\varepsilon^\sigma(x)|_{\varepsilon^\sigma} = 0$. Substituting (3.5) and (3.7)–(3.9) into (3.4) and integrating by parts the corresponding terms, after which we functionally differentiate the results with respect to $\varepsilon^\sigma(x)$, according to the boundary conditions of the functions $\varepsilon^\sigma(x)$, and set $J = K = 0$, we obtain (Li, 1999a)

$$\int D\phi D\pi \left\{ J^0_\sigma + \tilde{A}_{s\sigma}(z)\left(\frac{\delta I^P_{\text{eff}}}{\delta\phi_{(s)}(z)}\right) + \tilde{U}^s_\sigma(z)\left(\frac{\delta I^P_{\text{eff}}}{\delta\pi^{(s)}(z)}\right) \right. \\ -\tilde{R}^\mu_\sigma(z)\left[\phi_{(s)'\mu}(z)\frac{\delta I^P_{\text{eff}}}{\delta\phi_{(s)}(z)} + \pi^{(s)}_{,\mu}(z)\frac{\delta I^P_{\text{eff}}}{\delta\pi^{(s)}(z)}\right]$$

$$+\int d^4x\tilde{B}_{s\sigma}(z)\left[E(x,z)\frac{\delta I^P_{\text{eff}}}{\delta\phi_{(s)}(x)}+D(\pi^{(s)}(x)E(x,z))\right]$$
$$+\int d^4x\tilde{V}^s_\sigma\left[F(x,z)\frac{\delta I^P_{\text{eff}}}{\delta\pi^{(s)}(x)}\right]\Bigg\}\exp(iI^P)=0 \qquad (3.10)$$

where $J^0_\sigma=-i^{\delta\bar{J}[\phi_{(s)},\pi^{(s)},\varepsilon]}/\delta\varepsilon^\sigma(x)|_{\varepsilon^\sigma=0}$, and $\tilde{A}_{s\sigma}$, $\tilde{B}_{s\sigma}$, $\tilde{R}^\mu_\sigma$, $\tilde{U}^s_\sigma$ and $\tilde{V}^s_\sigma$ are adjoint operators with respect to $A_{s\sigma}$, $B_{s\sigma}$, R^μ_σ, U^s_σ and V^s_σ, respectively (Li, 1987). We deduce the expressions (3.10), the condition $\bar{J}[\phi_{(s)},\pi^{(s)},0]=1$ has been used. The Green function connected with (3.10) is given by

$$<0\Bigg|\Bigg\{J^0_\sigma+\tilde{A}_{s\sigma}(z)\left(\frac{\delta I^P_{\text{eff}}}{\delta\phi_{(s)}(z)}\right)+\tilde{U}^s_\sigma(z)\left(\frac{\delta I^P_{\text{eff}}}{\delta\pi^{(s)}(z)}\right)-\tilde{R}^\mu_\sigma(z)$$
$$\times\left(\phi_{(s)'\mu}(z)\frac{\delta I^P_{\text{eff}}}{\delta\phi_{(s)}(z)}+\pi^{(s)}_{,\mu}(z)\frac{\delta I^P_{\text{eff}}}{\delta\pi^{(s)}(z)}\right)+\int d^4x\tilde{B}_{s\sigma}(z)\left[E(x,z)\frac{\delta I^P_{\text{eff}}}{\delta\phi_{(s)}(x)}\right.$$
$$\left.+D(\pi^{(s)}(x)E(x,z))+\tilde{V}^s_\sigma F(x,z)\frac{\delta I^P_{\text{eff}}}{\delta\pi^{(s)}(x)}\right]\Bigg\}\Bigg|0\geq 0 \qquad (3.11)$$

where the symbol T^* stands for the covariantized T product (Young, 1987), and 0> is the vacuum state of the fields.

Substituting (3.5) and (3.7)–(9) into (3.4), and functionally differentiating the phase-space generating functional with respect to $\varepsilon^\sigma(x)$, we can obtain

$$\Bigg\{J^0_\sigma+\tilde{A}_{s\sigma}(z)\left(\frac{\delta I^P_{\text{eff}}}{\delta\phi_{(s)}(z)}\right)+\tilde{U}^s_\sigma(z)\left(\frac{\delta I^P_{\text{eff}}}{\delta\pi^{(s)}(z)}\right)-\tilde{R}^\mu_\sigma(z)\Big[\phi_{(s),\mu}(z)$$
$$\times\left(\frac{\delta I^P_{\text{eff}}}{\delta\phi_{(s)}(z)}+J^s(z)\right)+\pi^{(s)}_{,\mu}(z)\left(\frac{\delta I^P_{\text{eff}}}{\delta\pi^{(s)}(z)}+K_s(z)\right)\Big]$$
$$+\int d^4x\left[\tilde{B}_{s\sigma}(z)(E(x,z)\left(\frac{\delta I^P_{\text{eff}}}{\delta\phi_{(s)}(x)}+J(x)\right)+D(\pi^{(s)}(x)E(x,z))\right.$$
$$\left.+\tilde{V}^s_\sigma(x,z)(F(x,z)\left(\frac{\delta I^P_{\text{eff}}}{\delta\pi^{(s)}(x)}+K_s(x)\right)\right]\Bigg\}\Bigg|_{\substack{\phi_{(s)}\to\frac{1}{i}\frac{\delta}{\delta J^s}\\ \pi^{(s)}\to\frac{1}{i}\frac{\delta}{\delta K_s}}} Z[J,K]=0 \qquad (3.12)$$

Expression (3.12) can be called the generalized canonical Ward identities (GCWI) under the local and non-local transformation for a system with a singular higher-order Lagrangian in field throries. In the case for local transformation ($E=F=0$), and the Jacobian of the corresponding transformation is independent of $\varepsilon^\sigma(x)$, this implies that $J_\sigma^0 = 0$, from (3.12) we have

$$\begin{aligned}&\Big\{\tilde{A}_{s\sigma}\left(\frac{\delta I^P_{\text{eff}}}{\delta\phi_{(s)}}\right) - \tilde{R}^\mu_\sigma\left(\phi_{(s),\mu}\frac{\delta I^P_{\text{eff}}}{\delta\phi_{(s)}}\right) + \tilde{U}^s_\sigma\left(\frac{\delta I^P_{\text{eff}}}{\delta\pi^{(s)}}\right) - \tilde{R}^\mu_\sigma\left(\pi^{(s)}_{,\mu}\frac{\delta I^P_{\text{eff}}}{\delta\pi^{(s)}}\right)\\ &\tilde{A}_{s\sigma}J^s - \tilde{R}^\mu_\sigma(\phi_{(s),\mu}J^s) + \tilde{U}^s_\sigma K_s - \tilde{R}^\mu_\sigma(\pi^{(s)}_{,\mu}K_s)\Big\}\Big|_{\substack{\phi_{(s)}\to\frac{1}{i}\frac{\delta}{\delta J^s}\\ \pi^{(s)}\to\frac{1}{i}\frac{\delta}{\delta K_s}}} Z[J,K] = 0\end{aligned} \tag{3.13}$$

We functionally differentiate expressions (3.12) or (3.13) with respect to exterior source J, we can obtain another GCWI from (3.2). If one replaces $\pi^{(s)}$ by $\phi_{(s)}$ in (3.12) or (3.13), these GCWI can be expressed in terms of variables in configuration space, and some relations among Green functions can be obtained immediately in which one does not need to carry out the integration over the canonical momenta in phase-space generating functional (3.1).

4. GAUGE-INVARIANT SYSTEM

A system with a gauge-invariant Lagrangian $\mathcal{L}$ involving higher-order derivatives of the field variables, which is a generalized constrained canonical system (Li, 1994b), for this system, the effective Lagrangian $\mathcal{L}_{\text{eff}}$ in configuration space can be found by using the Faddeev–Popov trick through a transformation of the functional integral (Gitman and Tyutin, 1990),

$$\mathcal{L}_{\text{eff}} = \mathcal{L} + \mathcal{L}_f + \mathcal{L}_{gh},$$

where $\mathcal{L}_f$ is determined by the gauge conditions and L_{gh} is a ghost term. The configuration-space generating functional of the Green function for this system can be written as

$$Z[J] = \int \mathcal{D}\phi \exp\left\{i\int d^4x(\mathcal{L}_{\text{eff}} + J\phi)\right. \tag{4.1}$$

where ϕ represents all field variables, J represents all exterior sources.

Let us consider the transformation properties of configuration-space generating functional of the system under general local and non-local transformations, whose infinitesimal transformation is given by

$$\begin{cases} x^{\mu'} = x^\mu + \Delta x^\mu = x^\mu + R^\mu_\sigma \varepsilon^\sigma(x) \\ \phi(x') = \phi(x) + \Delta\phi(x) = \phi(x) + A_\sigma\varepsilon^\sigma(x) + \int d^4y E(x,y)B_\sigma\varepsilon^\sigma(y)\end{cases} \tag{4.2}$$

where $\varepsilon^\sigma(x)$ ($\sigma = 1, 2, \ldots, r$) are arbitrary infinitesimal independent functions; the values of $\varepsilon^\sigma(x)$ and their derivatives up to required order on the boundary of space-time domain vanish, and R^μ_σ, A_σ and B_σ are the linear differential operators. Under the transformation (4.2), the variation of the effective action

$I_{\mathrm{eff}} = \int d^4x \mathcal{L}_{\mathrm{eff}}$ is given by (Saito *et al.*, 1989)

$$\Delta I_{\mathrm{eff}} = \int d^4x \left\{ \frac{\delta I_{\mathrm{eff}}}{\delta\phi} \left(\left(A_\sigma - \phi_{,\mu} R_\sigma^\mu\right)\varepsilon^\sigma(x) + \int d^4y E(x,y) B_\sigma(y)\varepsilon^\sigma(y) \right) + \partial_\mu\left(j_\sigma^\mu \varepsilon^\sigma(x)\right) + \partial_\mu \left[\sum_{m=0}^{N-1} \prod^{\mu\nu(m)} \partial_{\nu(m)} \int d^4y E(x,y) B_\sigma(y)\varepsilon^\sigma(y) \right] \right\} \tag{4.3}$$

where

$$\frac{\delta I_{\mathrm{eff}}}{\delta\phi} = (-1)^m \partial_{\mu(m)} \mathcal{L}_{\mathrm{eff}}^{\mu(m)} \tag{4.4}$$

$$\mathcal{L}_{\mathrm{eff}}^{\mu(m)} = \frac{1}{m!} \sum_{\substack{\text{all permutation} \\ \text{of indices } \mu(m)}} \frac{\partial \mathcal{L}_{\mathrm{eff}}}{\partial \phi_{,\mu(m)}} \tag{4.5}$$

$$\prod^{\mu\nu(m)} = \sum_{l=0}^{N-(m+1)} (-1)^l \partial_{\lambda(l)} \mathcal{L}_{\mathrm{eff}}^{\mu\nu(m)\lambda(l)} \tag{4.6}$$

$$j_\sigma^\mu = \mathcal{L}_{\mathrm{eff}} R_\sigma^\mu + \sum_{m=0}^{N-1} \prod^{\mu\nu(m)} \partial_{\nu(m)} \left(A_\sigma - \phi_{,\lambda} R_\sigma^\lambda\right) \tag{4.7}$$

Using the Gauss theorem, we get the term of integral of $\partial_\mu(j_\sigma^\mu \varepsilon^\sigma)$ in (4.3) to vanish because of the boundary conditions of $\varepsilon^\sigma(x)$. It is supposed that the Jacobian of the transformation is equal to unity, substituting (4.2) and (4.3) into (4.1) and integrating by parts the corresponding terms, after which we functionally differentiate the results with respect to $\varepsilon^\sigma(z)(\sigma = 1, 2, ..., r)$, we obtain

$$\left\{ \tilde{A}_\sigma \left(\frac{\delta I_{\mathrm{eff}}}{\delta\phi(z)} + J \right) - \tilde{R}_\sigma^\mu \left[\phi_{,\mu} \left(\frac{\delta I_{\mathrm{eff}}}{\delta\phi(z)} + J \right) \right] + \int d^4x \tilde{B}_\sigma \left[E(x,z) \frac{\delta I_{\mathrm{eff}}}{\delta\phi(x)} + \partial_\mu \left(\sum_{m=0}^{N-1} \prod^{\mu\nu(m)} \partial_{\nu(m)} E(x,z) \right) + J \right] \right\} Z[J] = 0 \tag{4.8}$$

where $\tilde{A}_\sigma$, $\tilde{R}_\sigma^\mu$ and $\tilde{B}_\sigma$ are the adjoint operators with respect to A_σ, R_σ^μ and B_σ, respectively (Li, 1987). Expression (4.8) is called the generalized Ward identities under the general local and non-local transformation in configuration space for a gauge-invariant system with higher-order derivatives in field theories.

Let us now consider a global transformation in configuration space whose infinitesimal transformation is given by

$$\begin{cases} x^{\mu'} = x^\mu + \Delta x^\mu = x^\mu + \varepsilon_\sigma \tau^{\mu\sigma}(x, ..., \phi_{,\mu(m)}, ...) \\ \phi'(x') = \phi(x) + \Delta\phi(x) = \phi(x) + \varepsilon_\sigma \xi^\sigma(x, ..., \phi_{,\mu(m)}, ...) \end{cases} \tag{4.9}$$

where $\varepsilon_\sigma(\sigma = 1, 2, ..., r)$ are infinitesimal arbitrary parameters, $\tau^{\mu\sigma}$ and ξ^σ are some functions of x and $\phi_{,\mu(m)}$. It is supposed that effective action is invariant under the transformation (4.9) and the Jacobian of the transformation (4.9) is equal to unity. The configuration-space generating functional (4.1) is invariant under the transformation (4.9), thus, we have

$$Z[J] = \int \mathcal{D}\phi \left\{1 + i\varepsilon_\sigma \int d^4x[J(\xi^\sigma - \phi_{,\mu}\tau^{\mu\sigma}) + \partial_\mu(J\phi\tau^{\mu\sigma})]\right\} \times \exp\left\{i \int d^4x(L_{eff} + J\phi) = \{1 + i\varepsilon_\sigma \int d^4x\left[J\left(\xi^\sigma - \tau^{\mu\sigma}\partial_\mu \frac{\delta}{i\delta J}\right) + \partial_\mu\left(\tau^{\mu\sigma} J \frac{\delta}{i\delta J}\right)\right]\right\}\Bigg|_{\phi\to\frac{\delta}{i\delta J}} Z[J] = 0 \tag{4.10}$$

Consequently, we obtain the following result: If the effective action in configuration space is invariant under the transformation (4.8) and the Jacobian of this transformation is equal to unity, then the configuration-space generating functional of Green function satisfies the following identities

$$\int d^4x \left[J\left(\xi^\sigma - \tau^{\mu\sigma}\partial_\mu \frac{\delta}{i\delta J}\right) + \partial_\mu\left(\tau^{\mu\sigma} J \frac{\delta}{i\delta J}\right)\right]\Bigg|_{\phi\to\frac{\delta}{i\delta J}} Z[J] = 0 \tag{4.11}$$

The expression (4.11) is called generalized Ward identities for global symmetry transformation in configuration space.

For the internal symmetry transformation, $\tau^{\mu\sigma} = 0$, in this case, identities (4.11) can be written as

$$\int d^4x J\xi^\sigma \left(x, \frac{\delta}{i\delta J}\right) Z[J] = 0 \tag{4.12}$$

Functionally, differentiating (4.11) or (4.12) with respect to the exterior sources J many times and setting all exterior sources equal to zero, one can obtain various relationships among the Green function.

5. QUANTAL CONSERVATION LAWS

The canonical global symmetries of a dynamical system yield conservation laws in classical theories (Li, 1993b; Deriglazov and Evdokimov, 2000). Now let us consider a local transformation connected with the global symmetry transformation in phase space

$$x^{\mu'} = x^\mu + \Delta x^\mu = x^\mu + \varepsilon_\sigma(x)\tau^{\mu\sigma}\left(x, \phi_{(s)}, \pi^{(s)}\right), \tag{5.1a}$$

$$\phi'_{(s)}(x') = \phi_{(s)}(x) + \Delta\phi_{(s)}(x) = \phi_{(s)}(x) + \varepsilon_\sigma(x)U^\sigma_{(s)}\left(x, \phi_{(s)}, \pi^{(s)}\right), \tag{5.1b}$$

$$\pi^{(s)'}(x') = \pi^{(s)}(x) + \Delta\pi^{(s)}(x) = \pi^{(s)}(x) + \varepsilon_\sigma(x) + \varepsilon_\sigma(x)V^{(s)\sigma}\left(x, \phi_{(s)}, \pi^{(s)}\right), \tag{5.1c}$$

where $\varepsilon_\sigma(x)(\sigma = 1, 2, \cdots, r)$ are infinitesimal arbitrary functions and their values and derivatives vanish on the boundary of time–space domain. It is supposed that the effective canonical action in (3,4) is invariant under the global transformation in which the $\varepsilon_\sigma(x)$ in Equation (5.1) is replaced by arbitrary parameters ε_σ. Under the local transformation (5.1) the variation of the effective canonical action is given by

$$\begin{aligned}\Delta I^P_{\text{eff}} &= \int d^4x \Big\{ \frac{\delta I^P_{\text{eff}}}{\delta \phi_{(s)}} \big(\Delta\phi_{(s)} - \phi_{(s),\mu}\Delta x^\mu\big) + \frac{\delta I^P_{\text{eff}}}{\delta \pi^{(s)}} \big(\Delta\pi^{(s)} - \pi^{(s)}_{,\mu}\Delta x^\mu\big) \\ &\quad + \partial_\mu\big[\big(\pi^{(s)}\phi_{(s+1)} - \mathcal{H}_C\big)\Delta x^\mu\big] + D\big[\pi^{(s)}\big(\Delta\phi_{(s)} - \phi_{(s),\mu}\Delta x^\mu\big)\big]\Big\} \\ &= \int d^4x \varepsilon_\sigma(x) \Big\{ \frac{\delta I^P_{\text{eff}}}{\delta \phi_{(s)}} \big(U^\sigma_{(s)} - \phi_{(s),\mu}\tau^{\mu\sigma}\big) + \frac{\delta I^P_{\text{eff}}}{\delta \pi^{(s)}} \big(V^{(s)\sigma} - \pi^{(s)}_{,\mu}\Delta x^\mu\big) \\ &\quad + \partial_\mu\big[\big(\pi^{(s)}\phi_{(s+1)} - \mathcal{H}_C\tau^{\mu\sigma}\big)\big] + D\big[\pi^{(s)}\big(U^\sigma_{(s)} - \phi_{(s),\mu}\tau^{\mu\sigma}\big)\big]\Big\} \\ &\quad + \int d^4x \big\{\big[\big(\pi^{(s)}\phi_{(s+1)} - \mathcal{H}_C\big)\big]\tau^{\mu\sigma}\partial_\mu\varepsilon_\sigma(x) \\ &\quad + \pi^{(s)}\big(U^\sigma_{(s)} - \phi_{(s),\mu}\tau^{\mu\sigma}\big) D\varepsilon_\sigma(x)\big\} \end{aligned} \tag{5.2}$$

Because the effective canonical action is invariant under the global transformation, thus, the first integral in expression (5.2) is equal to zero. According to the boundary condition of $\varepsilon_\sigma(x)$, the expression (5.2) can be written as

$$\begin{aligned}\Delta I^P_{\text{eff}} = -\int d^4x \varepsilon_\sigma(x) \big\{ \partial_\mu \big[\big(\pi^{(s)}\phi_{(s+1)} - \mathcal{H}_C\big)\tau^{\mu\sigma}\big] \\ + D\big[\pi^{(s)}\big(U^\sigma_{(s)} - \phi_{(s),\mu}\tau^{\mu\sigma}\big)\big]\big\}\end{aligned} \tag{5.3}$$

The Jacobian of the local transformation (5.2) is denoted by $\overline{J}[\phi, \pi, \varepsilon]$.The phase-space generating function (3.4) is invariant under the local transformation (5.1), i.e. $\delta Z[J, K]/\delta\varepsilon_\sigma(x)|_{\varepsilon_\sigma=0} = 0$. Substituting Equations (5.1) and (5.3) into Equation (3.4) and functionally differentiating with respect to $\varepsilon_\sigma(x)$, we obtain

$$\begin{aligned}\int \mathcal{D}\varphi_{(s)}\mathcal{D}\pi^{(s)} \big\{\partial_\mu\big[\big(\pi^{(s)}\varphi_{(s)} - \mathcal{H}_{\text{eff}}\big)\tau^{\mu\sigma}\big] + D\big[\pi^{(s)}\big(U^\sigma_{(s)} - \varphi_{(s),\mu}\tau^{\mu\sigma}\big)\big] \\ - J^\sigma_0 - M^\sigma\big\} \exp\left[i\int d^4x\big(\mathcal{L}^P_{\text{eff}} + j^{(s)}\varphi_{(s)} + K_{(s)}\pi^{(s)}\big)\right] = 0\end{aligned} \tag{5.4}$$

where

$$J^\sigma_0 = -i\delta\bar{J}[\varphi, \pi, \varepsilon]/\delta\varepsilon_\sigma(x)|_{\varepsilon_\sigma(x)=0} \tag{5.5}$$

$$M^\sigma = J^{(s)}\big(U^\sigma_{(s)} - \varphi_{(s),\mu}\tau^{\mu\sigma}\big) + K_{(s)}\big(V^{(s)\sigma} - \pi^{(s)}_\mu\tau^{\mu\sigma}\big) \tag{5.6}$$

Functionally, differentiating (5.4) with respect to $J^{(0)}$ n times, one gets

$$\begin{aligned}&\int \mathcal{D}\varphi_{(s)}\mathcal{D}\pi^{(s)}\{\partial_\mu[(\pi^{(s)}\varphi_{s)} - \mathcal{H}_{\text{eff}})\tau^{\mu\sigma}]\\ &\quad + D\big[\pi^{(s)}\big(U^\sigma_{(s)} - \varphi_{(s),\mu}\tau^{\mu\sigma}\big)\big] - J^\sigma_0 - M^\sigma\}\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)\\ &\quad - i\sum_j \varphi(x_1)\varphi(x_2)\cdots\varphi(x_{j-1})\varphi(x_{j+1})\cdots\varphi(x_n)\\ &\quad \times N^\sigma\delta(x - x_j)\exp\left[i\int d^4x\big(\mathcal{L}^P_{\text{eff}} + J^{(s)}\varphi_{(s)} + K_{(s)}\pi^{(s)}\big)\right] = 0\end{aligned} \tag{5.7}$$

where

$$N^\sigma = U^\sigma_{(0)} - \varphi_{,\mu}\tau^{\mu\sigma} \tag{5.8}$$

Let us set all exterior sources equal to zero in expression (5.7); $J^{(s)} = K_{(s)} = 0$; we obtain

$$\begin{aligned}&\langle 0|T^*\{\partial_\mu[(\pi^{(s)}\dot{\varphi}_{(s)} - \mathcal{H}_{eff})\tau^{\mu\sigma}]\\ &\quad + D[\pi^{(s)}(U^\sigma_{(s)} - \varphi_{(s),\mu}\tau^{\mu\sigma})] - J^\sigma_0\}\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)|\,0\rangle\\ &= i\sum_j \langle 0|T^*[\varphi\ (x_1)\varphi(x_2)\cdots\varphi(x_{j-1})\varphi(x_{j+1})\cdots\varphi(x_n)N^\sigma|\,0\rangle\,\delta(x - x_j)\end{aligned} \tag{5.9}$$

where the symbol T^* stands for the convariantized T product (Young, 1987). Fixing t and letting

$$t_1, t_2, \cdots t_m \to +\infty, \qquad t_{m+1}, t_{m+2}, \cdots t_n \to -\infty$$

and using the reductin formula (12), we can write expression (5.9) as

$$\begin{aligned}&\langle \text{out}, m|\{\partial_\mu\big[\big(\pi^{(s)}\dot{\varphi}_{(s)} - \mathcal{H}_{\text{eff}}\big)\tau^{\mu\sigma}\big]\\ &\qquad + D\big[\pi^{(s)}U^\sigma_{(s)} - \varphi_{(s),\mu}\tau^{\mu\sigma}\big)\big] - J^\sigma_0\}|n - m, \text{in}\rangle = 0\end{aligned} \tag{5.10}$$

As m and n are arbitrary, this implies

$$\partial_\mu\left[\left(\pi^{(s)}\dot{\varphi}_{(s)} - H_{\text{eff}}\right)\tau^{\mu\sigma}\right] + D\left[\pi^{(s)}\left(U^\sigma_{(s)} - \varphi_{(s),\mu}\tau^{\mu\sigma}\right)\right] = J^\sigma_0 \tag{5.11}$$

It is supposed that the Jacobian of transformation (5.1) is a constant [or independent of $\varepsilon_\sigma(x)$]; in this case, $J^\sigma_0 = 0$. We take the integral of the expression (5.11) on three-dimensional space. If we assume that the fields have a configuration which vanishes rapidly at spatial infinity, according to Gauss' theorem, we obtain the following conserved quantity at the quantum level:

$$Q^\sigma = \int d^3x\left[\pi^{(s)}\left(U^\sigma_{(s)} - \varphi_{(s),k}\tau^{\mu\sigma}\right) - \mathcal{H}_{\text{eff}}\tau^{0\sigma}\right] \tag{5.12}$$

This result holds for anomaly-free theories.

The conservation law (5.12) in the quantum case corresponds to the classical conservation laws derived from the canonical Noether theorem (Li, 1993b; Deriglazov and Evdokimov, 2000). In general, H_{eff} differs from the canonical Hamiltonian H_C and the Jacobian of the transformation (5.1) may not be a constant; then the conserved quantity (5.12) is different from the classical ones. The connection between the symmetries and conservation laws in classical theories in general is no longer preserved in quantum theories.

The advantage of the aforementioned formalism for obtaining conservation laws at the quantum level is that we do not need to carry out explicit integration over the canonical momenta in the phase-space generating functional. In the general case it is difficult or impossible to carry out these integrations.

6. GAUGE INVARIANCE MASSIVE VECTOR FIELD WITH HIGHER-ORDER DERIVATIVES

Consider a massive vector field $B_\mu(x)$ with a scalar field $\phi(x)$ whose Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - c^2\partial_\lambda F^{\alpha\lambda}\partial_\rho F^\rho_\alpha + \frac{1}{2}(\partial_\mu\phi - mB_\mu)(\partial^\mu\phi - mB^\mu) \tag{6.1}$$

where the field strength tensor is expressed in terms of potentials in the usual way, $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, and c is a constant. The CS theory plays an important role in quantum Hall effect and high-T_C superconductivity. Superconductivity is characterized by a spontaneous breakdown of electromagnetic gauge invariance. It has been shown that the Lagrangian (6.1) for the case c=0 is functionally equivalent to mixed CS theory (Dorey and Mavomatos, 1990)

The Euler–Lagrange equations arising from (6.1) are

$$(1 - 2c^2\Box)\partial_\lambda F^\lambda_\mu - m^2B_\mu + m\partial_\mu\phi = 0 \tag{6.2}$$

where $\Box = g^{\mu\nu}\partial_\mu\partial_\nu$. In terms of the potentials the above equation reads

$$(1 - 2c^2\Box)\Box B_\mu - \partial_\mu[(1 - 2c^2\Box)\partial^\nu B_\nu] - m^2B_\mu + m\partial_\mu\phi = 0 \tag{6.3}$$

In passing to the Hamiltonian formalism we observe that the Lagrangian (6.1) is singular. The canonical momenta $\pi(x)$ conjugate to the scalar field $\phi(x)$ is

$$\pi = \frac{\delta L}{\delta(\partial_0\phi)} = -mB^0 + \dot{\phi}. \tag{6.4}$$

The canonical momenta $\pi_\mu(x)$ and $\pi^{(1)}_\mu(x)$ conjugate to the vector fields $B^\mu(x)$ and $B^\mu_{(1)}(x) = \dot{B}^\mu(x)$ are

$$\pi_\mu(x) = -F_{0\mu} - 2c^2\left(\partial_k\partial_\lambda F^{0\lambda}\delta^k_\mu - \partial_0\partial_\lambda F^\lambda_\mu\right) \tag{6.5}$$

$$\pi^{(1)}_\mu(x) = 2c^2\left(\partial_\lambda F^{0\lambda}\delta^0_\mu - \partial_\lambda F^\lambda_\mu\right) \tag{6.6}$$

and we get the primary constraint

$$\Phi^0 = \pi^{(1)}_0 \approx 0. \tag{6.7}$$

The canonical Hamiltonian is given by

$$H_C = \int_V d^3x \mathcal{H}_C = \int_V d^3x \Big[\pi_\mu B^\mu_{(1)} - \frac{1}{4c^2} \big(\pi^{(1)}_i\big)^2 + \pi_i \partial_k F^{ik} + \pi^{(1)}_i \partial^i B^0_{(1)}$$
$$+ \frac{1}{2} (B_{(1)i} - \partial_i B_0) \big(B^i_{(1)} - \partial^i B_0\big) - c^2 \left(\partial_i B^i_{(1)} - \partial_i \partial^i B_0\right) \left(\partial_k B^k_{(1)} - \partial_k \partial^k B_0\right)$$
$$+ F_{ij} F^{ij} + \frac{1}{2} m^2 B_i B^i + \frac{1}{2} \nabla\phi \cdot \nabla\phi - m B_i \partial^i \phi - (\partial^i \pi_i + m\pi) B^0 \tag{6.8}$$

The total Hamiltonian is given by

$$H_T = \int_V d^3x (\mathcal{H}_C + \lambda \Phi^0) \tag{6.9}$$

where $\lambda(x)$ is a Lagrangian multiplier. The stationarity for the constraints yields the following secondary constraints:

$$\Phi^1 = \{\Phi^0, H_T\} = \partial^i \pi^{(1)}_i - \pi_0 \approx 0 \tag{6.10}$$

$$\Phi^2 = \{\Phi^1, H_T\} = \partial^i \pi_i + m\pi \approx 0 \tag{6.11}$$

All the constraints $\Phi^k (k = 0, 1, 2)$ are first-class ones.

Let us take $\varepsilon_j(x) = \varepsilon(x)$ in (2.11), from (2.14), we have $G^j_3 = \Phi^0$, $G^j_2 = \Phi^1$, $G^j_1 = \Phi^2$, thus, the generator of gauge transformation can be constructed by using (2.11) to obtain

$$G = \int_V d^3x \big[\pi_\mu \partial^\mu \varepsilon(x) + m\pi \varepsilon(x) + \pi^{(1)}_\mu \partial_0 \partial^\mu \varepsilon(x)\big] \tag{6.12}$$

The gauge transformations induced by G are

$$\begin{cases} \delta B^\mu = \{B^\mu, G\} = \partial^\mu \varepsilon(x), \quad \delta B^\mu_{(1)} = \partial_0 \partial^\mu \varepsilon(x), \quad \delta\phi = m\varepsilon(x) \\ \delta\pi_\mu = \delta\pi^{(1)}_\mu = \delta\pi = 0 \end{cases} \tag{6.13}$$

The Lagrangian is gauge invariant under the transformation (6.13).

A method to incorporate constraints in the notion of Feynman path-integral, for each first-class constraint a gauge condition should be chosen. The purpose of introducing gauge conditions in a theory is to remove the gauge freedom. The gauge conditions must be preserved by the dynamical evolution of the system. The clue to finding a good set of gauge conditions is given by an analysis of the equation of motion (6.3). The zero component can be formally solved for $B_0(x)$ to give

$$B_0 = \frac{1}{(1 - 2c^2\Box)\nabla^2 + m^2} \partial_0 [(1 - 2c^2\Box)\nabla \cdot \vec{B} - m\phi] \tag{6.14}$$

Let us consider the generalized Coulomb gauge condition

$$(1 - 2c^2\Box)\nabla \cdot \vec{B} - m\phi = 0 \tag{6.15}$$

We see that the time preservation of this condition is equivalent to set $B_0 = 0$, whereas consistency requires that we set $\dot{B}_0 = 0$ as well. Hence, we are led to the following set of gauge conditions:

$$\Phi_1^G = B_{(1)0} \approx 0 \tag{6.16}$$

$$\Phi_2^G = (1 - 2c^2\Box)\nabla \cdot \vec{B} - m\phi \tag{6.17}$$

$$\Phi_3^G = B_0 \approx 0 \tag{6.18}$$

The full set of constraints (6.7), (6.10), (6.11) and gauge constraints (6.16), (6.17), (6.18), are now second-class ones. It is easy to check that the det$\{\Phi, \Phi\}$ is independent of field variables, where $\Phi = (\Phi^k, \Phi_m^G)$. Thus, one can omit this factor from the generating functional (3.1).

The phase-space generating functional of Green function for Lagrangian (6.1) can be written as

$$\begin{aligned} &Z[J_\mu, J, \xi^k, \xi_m] = \int \mathcal{D}B^\mu \mathcal{D}B_{(1)}^\mu \mathcal{D}\pi_\mu \mathcal{D}\pi_\mu^{(1)} \mathcal{D}\phi \mathcal{D}\pi \mathcal{D}\lambda_k \mathcal{D}\mu^m \\ &\exp\left\{i \int d^4x \left(L_{\text{eff}}^P + J_\mu B^\mu + J\phi + \xi^k \lambda_k + \xi_m \mu^m\right)\right\} \end{aligned} \tag{6.19}$$

where J_μ, J, ξ^k and ξ_m are the exterior sources with respect to B^μ, ϕ, λ_k and μ^m, respectively,

$$\mathcal{L}_{\text{eff}}^P = \mathcal{L} + \lambda_k \Phi^k + \mu^m \Phi_m^G \tag{6.20}$$

$$\mathcal{L}^P = \pi_\mu \dot{B}^\mu + \pi_\mu^{(1)} \dot{B}_{(1)}^\mu + \pi\dot{\phi} - \mathcal{H}_C. \tag{6.21}$$

Here we only introduce exterior sources for field variables, the expression (6.19) is called the generating functional of the "coordinate" Green function (Gitman and Tyutin, 1987).

The generating functional (6.19) and Lagrangian L^P is invariant under the transformation (6.13). The Jacobian of the transformation (6.13) is equal to unity. The canonical Ward identities (3.13) for this singular Lagrangian can be written as

$$\begin{aligned} &\left[-\partial_0 \frac{\delta}{\delta\xi_1} + \nabla^2(1 - 2c^2\Box - m^2)\frac{\delta}{\delta\xi_2} - \partial_0 \frac{\delta}{\delta\xi_3} - \partial^\mu J_\mu + mJ\right] \\ &\quad \times Z[J_\mu, J, \xi^k, \xi_m] = 0 \end{aligned} \tag{6.22}$$

As usual, let $Z[J_\mu, J, \xi^k, \xi_m] = \exp\{iW[J_\mu, J, \xi^k, \xi_m]\}$, and use the definition of generating functional of proper vertices $\Gamma[B^\mu, \phi, \lambda_k, \mu^m]$ which is given by performing a functional Legendre transformation

$$\Gamma[B^\mu, \phi, \lambda_k, \mu^m] = W[J_\mu, J, \xi^k, \xi_m] - \int d^4x(J_\mu B^\mu + J\phi + \xi^k \lambda_k + \xi_m \mu^m) \tag{6.23}$$

and

$$\frac{\delta W}{\delta} J_\mu(x) = B^\mu(x), \ \frac{\delta\Gamma}{\delta B^\mu(x)} = -J_\mu(x) \tag{6.24a}$$

$$\frac{\delta W}{\delta J(x)} = \phi(x), \quad \frac{\delta\Gamma}{\delta\phi(x)} = -J(x) \tag{6.24b}$$

$$\frac{\delta W}{\delta \xi^k(x)} = \lambda_k(x), \ \frac{\delta\Gamma}{\delta\lambda_k(x)} = -\xi^k(x) \tag{6.24c}$$

$$\frac{\delta W}{\delta \xi_m(x)} = \mu^m(x), \ \frac{\delta\Gamma}{\delta\mu^m(x)} = -\xi_m(x) \tag{6.24d}$$

Thus, the expression (6.22) becomes

$$-\partial_0\mu_1(x) - \nabla^2(1 - 2c^2\Box - m^2)\mu_2(x) + \partial_0\mu_3(x) + \partial_\mu \frac{\delta\Gamma}{\delta B_\mu(x)} - m\frac{\delta\Gamma}{\delta\phi(x)} = 0 \tag{6.25}$$

Functionally, differentiating (6.25) with respect to $\phi(x_2)$, and setting all fields (including multiplier fields) equal to zero, $\phi = B_\mu = \mu_1 = \mu_2 = \mu_3 = 0$, we obtain

$$\frac{\delta^2\Gamma[0]}{\delta\phi(x_1)\delta\phi(x_2)} = \frac{1}{m}\partial^\mu_{x_1}\left(\frac{\delta^2\Gamma[0]}{\delta B^\mu(x_1)\delta\phi(x_2)}\right) \tag{6.26}$$

This expression indicates that the propagator of field $\phi(x)$ should satisfy (6.26). Differentiation of (6.25) with respect to $B^\nu(x_2)$, and setting all fields equal to zero, yield

$$\partial^\mu_{x_1}\left(\frac{\delta^2\Gamma[0]}{\delta B^\mu(x_1)\delta B^\nu(x_2)}\right) = m\frac{\delta^2\Gamma[0]}{\delta\phi(x_1)\delta B^\nu(x_2)} \tag{6.27}$$

The propagator of massive vector field should satisfy (6.27). Differentiating (6.25) with respect to $B^\nu(x_2)$, and setting all fields equal to zero, we obtain

$$\partial^\mu_{x_1}\left(\frac{\delta^3\Gamma[0]}{\delta B^\mu(x_1)\delta B^\nu(x_2)\delta\phi(x_3)}\right) = m\frac{\delta^3\Gamma[0]}{\delta\phi(x_1)\delta B^\nu(x_2)\delta\phi(x_3)} \tag{6.28}$$

The expression (6.28) indicates that there are some relationships among the proper vertices. Similarly, differentiating (6.25) many times with respect to field variables one can obtain various Ward identities for proper vertices. Based on the canonical Ward identities to derive those relations has a significant advantage in that one does not need to carry out the integration over momenta as traditional treatment in configuration space.

The aforementioned results can also be derived by using the configuration-space generating functional given in Section 4.

7. NON-ABELIAN CHERN-SIMONS THEORIES WITH HIGHER-ORDER DERIVATIVES

The applications of canonical Ward identities in first-order derivative theories to Yang–Mills theory and other problems were given in previous work (Li, 1995, 1999b). Now we study the CS theories.

The CS term for both Abelian and non-Abelian cases has long been considered and is of increasing interest with direct applications to quantum Hall effect and high-T_C superconductivity. Now let us consider the $(2+1)$-dimensional non-Abelian CS fields A^a_μ coupled to the matter field ψ whose Lagrangian is given by

$$\mathcal{L} = -\frac{C^2}{4\pi} D_\rho F^a_{\mu\nu} D^\rho F^{a\mu\nu} - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{\kappa}{4\pi}\varepsilon^{\mu\nu\rho} \times \left(\partial_\mu A^a_\nu A^a_\rho + \frac{1}{3} f^a_{bc} A^a_\mu A^b_\nu A^c_\rho\right) + i\bar{\psi}\gamma^\mu D_\mu \psi \tag{7.1}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a_{bc} A^b_\mu A^c_\nu \tag{7.2}$$

and D_μ stands for the covariant derivative. The gauge invariance of non-Abelian CS term requires the quantization of the dimensionless constant $\kappa(\kappa = \frac{n}{4\pi}, (n \in Z))$ (Deser *et al.*, 1982). The Dirac γ-matrices are $\gamma^0 = \sigma^3$, $\gamma^1 = i\sigma^1$ and $\gamma^2 = i\sigma^2$ (σ's are the Pauli matrices).

According to the Ostrogradsky transformation, one can introduce the canonical momenta P^μ_a, Q^μ_a, $\bar{\pi}^{(\alpha)}_a(x)$ and $\pi^{(\alpha)}_a(x)$ with respect to A^a_μ, $\dot{A}^a_\mu = B^a_\mu$, $\psi_{(\alpha)}$ and $\bar{\psi}_{(\alpha)}$, respectively. The constraints in phase space are (Foussats *et al.*, 1995, 1996)

$$\theta^a_{(\alpha)} = \pi^a_{(\alpha)} \approx 0 \tag{7.3}$$

$$\bar{\theta}^a_{(\alpha)} = \bar{\pi}^a_{(\alpha)} + i(\bar{\psi}^a\gamma_0)_{(\alpha)} \approx 0 \tag{7.4}$$

$$\Lambda^{(0)a} = Q^{a0} \approx 0 \tag{7.5}$$

$$\Lambda^{(1)a} = -P^{a0} + D_i Q^{ai} \approx 0 \tag{7.6}$$

$$\Lambda^{(2)a} = f^a_{bc}(\bar{\psi}^b\pi^c + \bar{\pi}^b\psi^c) + D_i P^{ai} + \frac{\kappa}{4\pi}\varepsilon^{ij}\partial_i A^a_j + f^a_{bc} B^b_i Q^{ci} + f^a_{bc} A^b_0 D_i Q^{ci} \approx 0 \tag{7.7}$$

Equations (7.3) and (7.4) give the second-class constraints, and Equations (7.5), (7.6) and (7.7) give the first-class constraints. According to the rule of path-integral quantisation of a constrained canonical system, for each first-class constraint, a corresponding gauge condition should be chosen, and the gauge conditions are

(Foussats *et al.*, 1995, 1996)

$$\Omega_1^a = B_0^a \approx 0 \tag{7.8}$$

$$\Omega_2^a = \partial_i B^{ai} \approx 0 \tag{7.9}$$

$$\Omega_3^a = \partial_i A^{ai} \approx 0 \tag{7.10}$$

The phase-space generating functional for this model can be written as (Foussats *et al.*, 1995, 1996; Li, 1993a)

$$Z[J, \xi, \bar{\xi}] = \int \mathcal{D}u\delta(\Omega_i^a) \exp\left[i \int d^3x \left[\dot{A}_\mu^a P^{a\mu} + \dot{B}_\nu^a Q^{a\nu} + \dot{\bar{\psi}}\pi + \bar{\pi}\dot{\psi} - \mathcal{H}_C \right.\right.$$
$$+ \lambda_i^a \Lambda^{ia} + \bar{\lambda}_a^{(\alpha)}\theta_{(\alpha)}^a + \bar{\theta}_{(\alpha)}^a \lambda_a^{(\alpha)} - \partial^\mu \bar{C} D_{\mu b}^a C^b + J_a^\mu A_\mu^a + \bar{J}_{(\alpha)}\psi_{(\alpha)}$$
$$\left.\left. + \bar{\psi}_{(\alpha)} J_{(\alpha)} + \bar{\xi}_a C^a + \bar{C}^a \xi_a \right]\right] \tag{7.11}$$

where $\mathcal{H}_C$ is a canonical Hamiltonian density, $u = (A, B, \psi, \bar{\psi}, P, Q, \bar{\pi}, \pi, \lambda, \bar{\lambda}, C, \bar{C})$. Here we only introduce the exterior sources for field variables A_μ, ψ, $\bar{\psi}$, C^a and $\bar{C}^a$. The theory is independent of the choice of gauge conditions (Foussats *et al.*, 1995, 1996; Sundermeyer, 1982), $\Omega_i^a (i = 1, 2, 3)$ can be replaced by $\bar{\Omega}_i^a = \Omega_i^a - p_i^a(x)$, where $p_i^a(x)$ are independent of the gauge. Multiplying (7.11) by

$$\exp\left[-\frac{1}{2\alpha_i} \int d^3x \left(p_i^a\right)^2\right]$$

(α_i are parameters) and taking the path integral with respect to $p_i^a(x)$, we can obtain

$$Z[J, \xi, \bar{\xi}] = \int \mathcal{D}u \exp\left[i \int d^3x \left[\mathcal{L}_{eff}^{+} J_a^\mu A_\mu^a + \bar{J}_{(\alpha)}\psi_{(\alpha)}\right.\right.$$
$$\left.\left. + \bar{\psi}_{(\alpha)} J_{(\alpha)} + \bar{\xi}_a C^a + \bar{C}^a \xi_a \right]\right] \tag{7.12}$$

where

$$\mathcal{L}_{\text{eff}}^P = \mathcal{L}^P + \mathcal{L}_g + \mathcal{L}_{gh} + \mathcal{L}_m \tag{7.13}$$

$$\mathcal{L}^P = \dot{A}_\mu^a P^{a\mu} + \dot{B}_\nu^a Q^{a\nu} + \dot{\bar{\psi}}\pi + \bar{\pi}\dot{\psi} - \mathcal{H}_C \tag{7.14}$$

$$\mathcal{L}_g = \frac{1}{2\alpha_2}\left(\Omega_2^a\right)^2 = -\frac{1}{2\alpha_2}(\partial_\mu A^{a\mu})^2 \tag{7.15}$$

$$\mathcal{L}_{gh} = -\partial^\mu \bar{C}^a D_{b\mu}^a C_b$$

$$\mathcal{L}_m = \lambda_i^a \Lambda^{ia} + \bar{\lambda}_a^{(\alpha)} \theta_{(\alpha)}^a + \bar{\theta}_{(\alpha)}^a \lambda_a^{(\alpha)} - \frac{1}{2\alpha_1} \left(\Omega_1^a\right)^2 - \frac{1}{2\alpha_3} \left(\Omega_3^a\right)^2 \tag{7.16}$$

It is easy to check that the action connected with the terms L^P and L_{gh} in the theory is invariant under the following transformation (Li, 1995; Li, 1999b; Kuang and Yi, 1980)

$$C^{a'}(x) = C^a(x) + i(T_\sigma)_b^a C^b(x) \varepsilon^\sigma(x) \tag{7.17a}$$

$$\bar{C}^{a'}(x) = \bar{C}^a(x) - i\bar{C}^b(x)(T_\sigma)_b^a \varepsilon^\sigma(x) + \frac{i}{\Box} \partial_\mu [\bar{C}^b(x)(T_\sigma)_b^a \partial^\mu \varepsilon^\sigma(x)] \tag{7.17b}$$

$$A_\mu^{a'}(x) = A_\mu^a(x) + D_{\sigma\mu}^a \varepsilon^\sigma(x),\ A_{(1)\mu}^{a'} = A_{(1)\mu}^a + \partial_0 D_{\sigma\mu}^a \varepsilon^\sigma(x) \tag{7.17c}$$

$$\psi'(x) = \psi(x) - i(T_\sigma)\psi(x)\varepsilon^\sigma(x),\ \bar{\psi}'(x) = \bar{\psi}(x) + i\bar{\psi}(x)(T_\sigma)\varepsilon^\sigma(x) \tag{7.17d}$$

where $T_\sigma(\sigma = 1, 2, ..., r)$ are the generators of gauge group. Equation (7.17b) can be written as

$$\bar{C}^{a'}(x) = \bar{C}^a(x) - ig\bar{C}^b(x)(T_\sigma)_b^a \varepsilon^\sigma(x) + i \int \{d^3 y \Delta_0(x, y)\} \partial_\mu [\bar{C}^b(y)(T_\sigma)_b^a \partial^\mu \varepsilon^\sigma(y)] \tag{7.18a}$$

where

$$\Box \Delta_0(x - y) = i\delta^{\{}(2)\}(x - y) \tag{7.18b}$$

The Jacobian of the transformation (7.17) is denoted by $J_\varepsilon[\phi, \pi, \varepsilon]$. The quantities

$$J_\sigma^0 = \left. \frac{\delta J_\varepsilon}{\delta \varepsilon^\sigma} \right|_{\varepsilon^\sigma = 0}$$

are independent of the field variables (Li, 1995; Li, 1999b; Kuang and Yi, 1980). Let the change of $\mathcal{L}_g + \mathcal{L}_m$ is denoted by

$$\delta(\mathcal{L}_g + \mathcal{L}_m) = F_\sigma(u)\varepsilon^\sigma(x) \tag{7.19}$$

under the transformation (7.17), the invariance of generating functional (7.12) under the transformation (7.17) implies

$$\left. \frac{\delta Z}{\delta \varepsilon^\sigma} \right|_{\varepsilon^\sigma = 0} = 0.$$

Thus, one has the generalized Ward identities

$$\Big\{ J_\sigma^0 + iF_\sigma - i\partial_\mu J_\sigma^\mu + f_{\sigma c}^a J_a^\mu \frac{\delta}{\delta J_c^\mu} + i\bar{J}_\alpha (T_\sigma)_\beta^\alpha \frac{\delta}{\delta \bar{J}_\beta} - iJ_\alpha (T_\sigma)_\beta^\alpha \frac{\delta}{\delta J_\beta} + i\bar{\xi}_a (T_\sigma)_b^a \frac{\delta}{\delta \bar{\xi}_b} - i\xi_a (T_\sigma)_b^a \frac{\delta}{\delta \xi_b} + i\partial^\mu \Big[\partial_\mu \Big(\xi_a \frac{1}{\Box}\Big)(T_\sigma)_b^a \frac{\delta}{\delta \xi_b}\Big]\Big\} Z[J, \xi, \bar{\xi}] = 0 \tag{7.20}$$

Let $Z[J, \xi, \bar{\xi}] = \exp\{iW[J, \xi, \bar{\xi}]\}$ and use the definition of generating functional of proper vertices $\Gamma[Q, C, \bar{C}]$ which is given by performing a functional Legendre transformation of $W[J, \xi, \bar{\xi}]$. Then, the generalized Ward identities (7.20) can be written as

$$\begin{aligned} J_\sigma^0 + iF_\sigma + i\partial_\mu \tfrac{\delta\Gamma}{\delta A_\mu^\sigma} - if_{\sigma c}^a A_\mu^c \tfrac{\delta\Gamma}{\delta A_\mu^a} - i\psi_\beta (T_\sigma)_\beta^\alpha \tfrac{\delta\Gamma}{\delta\psi_\alpha} + i\bar{\psi}_\beta (T_\sigma)_\beta^\alpha \tfrac{\delta\Gamma}{\delta\bar{\psi}_\alpha} \\ -iC^a (T_\sigma)_b^a \tfrac{\delta\Gamma}{\delta C^b} + i\bar{C}^a (T_\sigma)_b^a \tfrac{\delta\Gamma}{\delta\bar{C}^b} - i\partial^\mu \left[\partial_\mu \left(\tfrac{\delta\Gamma}{\delta\bar{C}^a}\tfrac{1}{\Box}\right)(T_\sigma)_b^a \bar{C}^b\right] = 0 \end{aligned} \tag{7.21}$$

We functionally differentiate (7.21) with respect to $\psi_\delta(x_2)$, $\bar{\psi}_\rho(x_3)$ and set all fields equal zero $A_\mu^a = \psi = \bar{\psi} = C^a = \bar{C}^a = \lambda = 0$, because the quantities J_σ^0 are independent of field variables (Li, 1995; Li, 1999b; Kuang and Yi, 1980). Thus, we obtain the following relations:

$$\begin{aligned} \partial_{x_1}^\mu \frac{\delta^3\Gamma[0]}{\delta\bar{\psi}_\rho(x_3)\delta\psi_\delta(x_2)\delta A_\sigma^\mu(x_1)} &= \delta(x_1 - x_2)(T_\sigma)_\delta^\alpha \frac{\delta^2\Gamma[0]}{\delta\bar{\psi}_\rho(x_3)\delta\psi_\alpha(x_1)} \\ &\quad - \delta(x_1 - x_3)(T_\sigma)_\rho^\alpha \frac{\delta^2\Gamma[0]}{\delta\psi_\delta(x_2)\delta\bar{\psi}_\alpha(x_1)} \end{aligned} \tag{7.22}$$

We functionally differentiate (7.21) with respect to $\bar{C}^k(x_2)$ and $C^m(x_3)$ and set $B = \psi = \bar{\psi} = C = \bar{C} = \lambda = 0$, thus, we obtain

$$\begin{aligned} &\partial_{x_1}^\mu \frac{\delta^3\Gamma[0]}{\delta\bar{C}^k(x_2)\delta C^m(x_3)\delta A_\sigma^\mu(x_1)} - \partial^\mu \left[\partial_\mu \left(\frac{\delta^2\Gamma[0]}{\delta\bar{C}^a(x_1)\delta C^m(x_3)}\frac{1}{\Box}\right)(T_\sigma)_a^k \delta(x_1 - x_2)\right] \\ &\quad - \delta(x_1 - x_3)(T_\sigma)_b^m \frac{\delta^2\Gamma[0]}{\delta\bar{C}^k(x_2)\delta C^b(x_1)} + \delta(x_1 - x_2)(T_\sigma)_k^b \frac{\delta^2\Gamma[0]}{\delta\bar{C}^b(x_1)\delta C^m(x_3)} = 0 \end{aligned} \tag{7.23a}$$

Expression (7.23a) can also be written as

$$\begin{aligned} &\partial_{x_1}^\mu \frac{\delta^3\Gamma[0]}{\delta\bar{C}^k(x_2)\delta C^m(x_3)\delta A_\sigma^\mu(x_1)} + \partial_{x_1}^\mu \left[\partial_\mu^{x_1} \int d^3y \frac{\delta^2\Gamma[0]}{\delta\bar{C}^a(x_1)\delta C^m(x_3)} \right. \\ &\quad \left. \times \Delta_0(x_1 - y)(T_\sigma)_a^k \cdot \delta(x_1 - x_2)\right] - \delta(x_1 - x_3)(T_\sigma)_b^m \frac{\delta^2\Gamma[0]}{\delta\bar{C}^k(x_2)\delta C^b(x_1)} \\ &\quad + \delta(x_1 - x_2)(T_\sigma)_k^b \frac{\delta^2\Gamma[0]}{\delta\bar{C}^b(x_1)\delta C^m(x_3)} = 0 \end{aligned} \tag{7.23b}$$

Differentiating (7.21) many times with respect to field variables, one can obtain various Ward identities for proper vertices and propagators.

This formulation to derive the Ward identities for proper vertices has a significant advantage in that one does not need to carry out the integration over momenta in phase-space generating functional. The expression (7.23) is a new form of the

Ward identities of gauge-ghost proper vertices for CS theories which differs from the usual Ward–Takahashi identities arising from the BRS invariance. The usual BRS transformation is non-linear in the ghost fields, we present here that the transformation (7.17) is a linear one (but non-local). We derive the above relations for proper vertices in which the invariance of L^P and L_{gh} are only required, this is also different from the traditional treatment. The Ward identities in first-order derivatives theories for non-local transformation in configuration space was first discussed by Kuang and Yi (1980) from another point of view, and is useful to simplify the calculation in QCD (Kuang and Yi, 1980). The further application of these Ward identities in CS theories is in progress.

Let us now consider global transformation (for example, BRS transformation) . Consider the BRS transformation in the configuration space (where τ is a Grassmann's parameter),

$$\begin{cases} \delta A_\mu^a = -\tau D_{\mu b}^a C^b, & \delta B_\mu^a = -\tau \partial_0 D_{\mu b}^a C^b, \\ \delta\psi = i\tau C^b T^b \psi, & \delta\bar{\psi} = -i\tau\bar{\psi} C^b T^b, \\ \delta C^a = \frac{1}{2}\tau f_{bc}^a C^b C^c, & \delta\bar{C}^a = -\frac{\tau}{\alpha_2}\partial^\mu A_\mu^a \end{cases} \tag{7.24}$$

It is easy to check that $\delta(D_{b\mu}^a C^b) = \delta(\delta\psi) = \delta(\delta\bar{\psi}) = \delta(\delta C^a) = 0$ under the BRS transformation. We introduce exterior sources $u_a^\mu, v_a, \bar{\eta}$ and η with respect to $\delta A_\mu^a, \delta C^a, \delta\psi$ and $\delta\bar{\psi}$, respectively, and give the configuration-space extended functional as Expression (7.23a) can also be written as

$$\begin{aligned} Z[J,\xi,\bar{\xi},u,v,\eta,\bar{\eta}] = \int \mathcal{D}A_\mu^a \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\bar{C}^a\mathcal{D}C^a \exp\{i\textstyle\int d^3x(\mathcal{L}_{\rm eff} + J_a^\mu A_\mu^a \\ +\bar{J}\psi + \bar{\psi}J + \bar{\xi}_a C^a + \bar{C}^a\xi_a + u_\mu^a\delta A_\mu^a + v_a\delta C^a + \bar{\eta}\delta\psi + \delta\bar{\psi}\eta)\} \end{aligned} \tag{7.25}$$

where $\mathcal{L}_{\rm eff}$ is an effective Lagrangian in configuration space obtained by using the Faddeev–Popov trick under the Lorentz gauge. This effective Lagrangian and generating functional (7.25) is invariant under the transformation (7.24), the Jacobian of this transformation is equal to unity. Thus, we have

$$\begin{aligned} &\textstyle\int \mathcal{D}A_\mu^a\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\bar{C}^a\mathcal{D}C^a[\int d^3x(J_a^\mu\delta A_\mu^a + \bar{J}\delta\psi + \delta\bar{\psi}J + \bar{\xi}\delta C + \delta\bar{C}\xi)] \\ &\quad\times \exp\{i\textstyle\int d^3x(\mathcal{L}_{\rm eff} + J_a^\mu A_\mu^a + \bar{J}\psi + \bar{\psi}J + \bar{\xi}C + \bar{C}\xi \\ &\quad+ u_\mu^a\delta A_\mu^a + v_a\delta C^a + \bar{\eta}\delta\psi + \delta\bar{\psi}\eta)\} = 0 \end{aligned} \tag{7.26}$$

Consequently, we obtain the generalized Ward identity of the generating functional of Green function for the system with Lagrangian (7.1),

$$\begin{aligned} \int d^3x\left[\bar{J}\frac{\delta}{\delta\bar{\eta}} + J\frac{\delta}{\delta\eta} + J_a^\mu\frac{\delta}{\delta u_a^\mu} + \bar{\xi}_a\frac{\delta}{\delta v_a} - \partial_\mu\left(\frac{\delta}{\delta J_a^\mu}\right)\xi_a\right] \\ Z[J,\xi,\bar{\xi},u,v,\eta,\bar{\eta}] = 0 \end{aligned} \tag{7.27}$$

This result can also be derived by using the generating functional (7.12) in phase space. In addition, the effective Lagrangian is invariant under the BRS transformation, and the Jacobian of the BRS transformation is equal to unity; from (5.12) we can obtain the BRS conserved quantity for non-Abelian higher-derivative CS theories at the quantum level

$$Q = \int d^2x \left(\pi_a^\mu \delta A_\mu^a + Q_a^\mu \delta B_\mu^a + \bar{\pi}\delta\psi + \delta\bar{\psi}\pi + \bar{R}_a \delta C^a + \delta\bar{C}^a R_a\right) \qquad (7.28)$$

where $\bar{R}_a$ and R_a are the canonical momenta conjugate to C^a and $\bar{C}^a$, respectively (Li, 1997a).

8. CONCLUSIONS AND DISCUSSION

Dynamical system described in terms of higher-order derivatives Lagrangian has close relation with the modern field theory, and it has attracted much attention recently. Local gauge invariance is a fundamental concept in modern field theories. Gauge theories belong to the class of the so-called singular Lagrangian theories. A system with a singular Lagrangian is subject to some phase space inherent constraints. Here the symmetries in a constrained canonical system with a singular higher-order Lagrangian has been studied. First of all, we develop a simple algorithm for constructing the generator of gauge transformation of such a system, once the canonical Hamiltonian and first-class constraints are determined, the generator can be constructed. In the theory of path integral quantization for a dynamical system, the phase-space path integral has more basic sense. Based on the phase-space generating functional of the Green function for a system with a singular higher-order Lagrangian, the generalized canonical Ward identities for local and non-local transformation in extended phase space have been deduced. The quantal conservation laws have been derived. The advantage of this formalism is that one does not need to carry out the integration over canonical momenta in phase-space path integral as was done usually. In general case, especially for a constrained canonical system with complicated constraints, it is very difficult or even impossible to carry out the integration over canonical momenta. But for a gauge-invariant system one can use Faddeev–Popov trick to obtain the configuration-space generating functional., based on this generating functional, the generalized Ward identities in configuration space have been also derived for the local and non-local transformation. The applications of above results to the massive vector field and non-Abelian CS theories with higher-order derivatives are given. BRS symmetry at the quantum level is discussed.

Numerous recent studies of $(2+1)$-dimensional gauge theories with CS terms in Lagrangian have revealed the occurrence of fractional spin and statistics (Banerjee, 1993; Kim *et al.*, 1994). In those paper the angular-momentum was deduced by using classical Noether theorem. It has been pointed out that the results are valid at the quantum level in Abelian CS theories (Li, 1997b). In the non-Abelian CS theories the angular momentum at the quantum level is different from the classical one in that one needs to take into account the contribution of angular momenta of ghost fields (Li, 1997b), from (7.13) one can easy see that this result also hold true for non-Abelian higher-derivative CS theories (Li, 1997a). We do not think the conclusion (Antillon *et al.*, 1995; Banerjee and Chakraborty, 1996) in classical non-Abelian CS theories is valid at the quantum level.

The use of higher-order derivative terms in the Lagrangian allow us to improve the behavior of the corresponding propagators at large momentum, rendering the theory less divergent. The unitarity may be violated in higher-order derivative theories which needs further to study (Foussats *et al.*, 1995, 1996; Hauking, 1987).

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